MATH3210 - SPRING 2024 - SECTION 004

HOMEWORK 7 - *SOLUTIONS*

Problem 1 (20 points)**.** Prove that if f is defined on (a, b) is differentiable at c, $f(c) \neq 0$, and $g(x) := 1/f(x)$, then $g'(c) = -\dfrac{f'(c)}{f(c)^2}$.

Solution. Note that

$$\lim_{x\to c} \frac{g(x) - g(c)}{x - c} = \lim_{x\to c} \frac{1/f(x) - 1/f(c)}{x - c} = \lim_{x\to c} -\frac{1}{f(x)f(c)} \cdot \frac{f(x) - f(c)}{x - c}$$

Since f is differentiable at c, it is continuous at c, and since $f(c) \neq 0$, the limit of $\dfrac{1}{f(x)f(c)}$ as $x \to c$ is $\dfrac{1}{f(c)^2}$. It follows from the definition of limits that

$$g'(c) = -\frac{f'(c)}{f(c)^2}.$$

□

Problem 2 (80 points)**.** For each, either calculate $f'(0)$ with justification, or prove that f is not differentiable at 0. You may assume continuity and the usual properties and formulas for the function sin. [*Hints*: Try to sketch a graph if you can to get an idea. The points $x_n = 1/(2\pi n)$ are especailly useful in the graph and proofs for (c) and (d). The squeeze theorem is useful!]

(a) $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$

(c) $h(x) = \begin{cases} 0, & x = 0 \\ x\sin(1/x), & \text{otherwise} \end{cases}$

(d) $k(x) = \begin{cases} 0, & x = 0 \\ x^2\sin(1/x), & \text{otherwise} \end{cases}$

Solution. (a) We claim that f is differentiable at 0. It suffices to show that the left- and right-hand limits of $\dfrac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$ exist and are equal. Note that

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^2}{x} = 0$$

and

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{0}{x} = 0.$$

(b) We claim that the g is not differentiable at 0. As in (a), we will use the left- and right-hand limits, showing that they are *not* equal. We compute

$$\lim_{x\to 0^+} \frac{g(x)}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

and

$$\lim_{x\to 0^-} \frac{g(x)}{x} = \lim_{x\to 0^-} \frac{0}{x} = 0.$$

(c) We claim that h is not differentiable at 0. Indeed, if h was differentiable at 0, then $\lim_{x\to 0} \frac{h(x)}{x}$ exists. Since $x_n = 1/(\pi n + \pi/2)$ converges to 0, it follows if h were differentiable we would require that $\lim_{n\to\infty} \frac{h(x_n)}{x_n}$ exists. But

$$\frac{h(x_n)}{x_n} = \frac{x_n \sin(1/x_n)}{x_n} = \sin(\pi n + \pi/2) = (-1)^{n+1}.$$

Since this sequence does not converge, h cannot be differentiable at 0.

(d) We claim that k is differentiable at 0. Indeed,

$$\lim_{x\to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x\to 0} x \sin(1/x) = 0.$$

This limit exists by the squeeze theorem, since $|x\sin(1/x) - 0| = |x\sin(1/x)| \le |x| \to 0$.

$\square$